

# Flag-transitive point-primitive non-symmetric $2-(v, k, 2)$ designs with alternating socle\*

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March 3, 2016

## Abstract

This paper studies flag-transitive point-primitive non-symmetric  $2-(v, k, 2)$  designs. We prove that if  $\mathcal{D}$  is a non-trivial non-symmetric  $2-(v, k, 2)$  design admitting a flag-transitive point-primitive automorphism group  $G$  with  $\text{Soc}(G) = A_n$  for  $n \geq 5$ , then  $\mathcal{D}$  is a  $2-(6, 3, 2)$  or  $2-(10, 4, 2)$  design.

**MR(2000) Subject Classification:** 05B05, 05B25, 20B25

**Keywords:** primitive group; flag-transitive; non-symmetric design; alternating socle

## 1 Introduction

A  $2-(v, k, \lambda)$  *design* is a finite incidence structure  $\mathcal{D}=(\mathcal{P}, \mathcal{B})$  consisting of  $v$  points and  $b$  blocks such that every block is incident with  $k$  points, every point is incident with  $r$  blocks, and any two distinct points are incident with exactly  $\lambda$  blocks. The design  $\mathcal{D}$  is called *symmetric* if  $v = b$  (or equivalently  $r = k$ ) and *non-trivial* if  $1 < k < v$ . A *flag* of  $\mathcal{D}$  is an incident point-block pair  $(\alpha, B)$  where  $\alpha$  is a point and  $B$  is a block. An automorphism of  $\mathcal{D}$  is a permutation of the points which also permutes the blocks. The set of all automorphisms of  $\mathcal{D}$  with the composition of maps is a group, denoted by  $\text{Aut}(\mathcal{D})$ . A subgroup  $G \leq \text{Aut}(\mathcal{D})$  is called *point-primitive* if it acts primitively on  $\mathcal{P}$  and *flag-transitive* if it acts transitively on the set of flags of  $\mathcal{D}$ .

It is shown in [10] that the socle of the automorphism group of a flag-transitive, point-primitive symmetric  $2-(v, k, 2)$  design cannot be alternating or sporadic. Motivated by this article, it is natural to consider the case of non-symmetric designs. Recently, we proved in [7] that, for a non-symmetric  $2-(v, k, 2)$  design, if  $G \leq \text{Aut}(\mathcal{D})$  is flag-transitive and point-primitive then  $G$  must be an affine or almost simple group. Moreover, if the socle of  $G$  is

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\*This work is supported by the National Natural Science Foundation of China (Grant No.11471123).

sporadic, then  $\mathcal{D}$  is the unique  $2$ -(176, 8, 2) design with  $G = HS$ , the Higman-Sims simple group. This article is a continuation of [7]. Here we treat almost simple groups in which  $\text{Soc}(G)$  is an alternating group. The main result is the following.

**Theorem 1.1.** *If  $\mathcal{D}$  is a non-trivial non-symmetric  $2$ -( $v, k, 2$ ) design admitting a flag-transitive point-primitive automorphism group  $G$  with alternating socle  $A_n$  for  $n \geq 5$ , then*

- (i)  $\mathcal{D}$  is a unique  $2$ -(6, 3, 2) design and  $G = A_5$ , or
- (ii)  $\mathcal{D}$  is a unique  $2$ -(10, 4, 2) design and  $G = A_5, A_6$  or  $S_6$ .

The structure of our paper is as follows. In Section 2, we give some preliminary lemmas on flag-transitive designs and permutation groups. In Section 3, we complete the proof of Theorem 1.1 in 5 parts.

## 2 Preliminaries

**Lemma 2.1.** *The parameters  $v, b, r, k, \lambda$  of a non-trivial  $2$ -( $v, k, \lambda$ ) design satisfy the following arithmetic conditions:*

- (i)  $vr = bk$ ;
- (ii)  $\lambda(v - 1) = r(k - 1)$ ;
- (iii)  $b \geq v$  and  $k \leq r$ .

In particular, if the design is non-symmetric then  $b > v$  and  $k < r$ .

**Lemma 2.2.** *Let  $\mathcal{D}$  be a non-trivial  $2$ -( $v, k, \lambda$ ) design. Let  $\alpha$  be a point of  $\mathcal{D}$  and  $G$  be a flag-transitive automorphism group of  $\mathcal{D}$ .*

- (i)  $r^2 > \lambda v$  and  $|G_\alpha|^3 > \lambda|G|$ . In particular,  $r^2 > v$ .
- (ii)  $r \mid \lambda(v - 1, |G_\alpha|)$ , where  $G_\alpha$  is the stabilizer of  $\alpha$ .
- (iii) If  $d$  is any non-trivial subdegree of  $G$ , then  $r \mid \lambda d$  (and so  $\frac{r}{(r, \lambda)} \mid d$ ).

**Proof.** (i) The equality  $r = \frac{\lambda(v-1)}{k-1}$  implies  $\lambda v = r(k-1) + \lambda \leq r(r-1) + \lambda = r^2 - r + \lambda$ , and the non-triviality of  $\mathcal{D}$  implies  $r > \lambda$ , and so  $r^2 > \lambda v$ . Combining this with  $v = |G : G_\alpha|$  and  $r \leq |G_\alpha|$  by the flag-transitivity of  $G$ , we have  $|G_\alpha|^3 > \lambda|G|$ . (ii) Since  $G$  is flag-transitive and  $\lambda(v-1) = r(k-1)$ , we have  $r \mid \lambda(v-1)$  and  $r \mid |G_\alpha|$ . It follows that  $r$  divides  $(\lambda(v-1), |G_\alpha|)$ , and hence  $r \mid \lambda(v-1, |G_\alpha|)$ . Part (iii) was proved in [2, p.91] and [3].

□

**Lemma 2.3.** ([8, p.366]) *If  $G$  is  $A_n$  or  $S_n$ , acting on a set  $\Omega$  of size  $n$ , and  $H$  is any maximal subgroup of  $G$  with  $H \neq A_n$ , then  $H$  satisfies one of the following:*

- (i)  $H = (S_\ell \times S_m) \cap G$ , with  $n = \ell + m$  and  $\ell \neq m$  (intransitive case);
- (ii)  $H = (S_\ell \wr S_m) \cap G$ , with  $n = \ell m$ ,  $\ell > 1$  and  $\ell \neq m$  (imprimitive case);
- (iii)  $H = \text{AGL}_m(p) \cap G$ , with  $n = p^m$  and  $p$  a prime (affine case);
- (iv)  $H = (T^m \cdot (\text{Out } T \times S_m)) \cap G$ , with  $T$  a nonabelian simple group,  $m \geq 2$  and  $n = |T|^{m-1}$  (diagonal case);
- (v)  $H = (S_\ell \wr S_m) \cap G$ , with  $n = \ell^m$ ,  $\ell \geq 5$  and  $m > 1$  (wreath case);
- (vi)  $T \trianglelefteq H \leq \text{Aut}(T)$ , with  $T$  a nonabelian simple group,  $T \neq A_n$  and  $H$  acting primitively on  $\Omega$  (almost simple case).

**Remark 1.** This lemma does not deal with the groups  $M_{10}$ ,  $\text{PGL}_2(9)$  and  $\text{P}\Gamma\text{L}_2(9)$  that have  $A_6$  as socle. These exceptional cases will be handled in the first part of Section 3.

**Lemma 2.4.** ([9, Theorem (b)(I)]) *Let  $G$  be a primitive permutation group of odd degree  $n$  on a set  $\Omega$  with simple socle  $X = \text{Soc}(G)$ , and let  $H = G_\alpha$ ,  $\alpha \in \Omega$ . If  $X \cong A_c$ , an alternating group, then one of the following holds:*

- (i)  $H$  is intransitive, and  $H = (S_a \times S_{c-a}) \cap G$  where  $1 \leq a < \frac{1}{2}c$ ;
- (ii)  $H$  is transitive and imprimitive, and  $H = (S_a \wr S_{c/a}) \cap G$  where  $a > 1$  and  $a \mid c$ ;
- (iii)  $H$  is primitive,  $n = 15$  and  $G \cong A_7$ .

**Lemma 2.5.** ([5, Theorem 5.2A]) *Let  $G = \text{Alt}(\Omega)$  where  $n = |\Omega| \geq 5$ , and let  $s$  be an integer with  $1 \leq s \leq \frac{n}{2}$ . Suppose that,  $K \leq G$  has index  $|G : K| < \binom{n}{s}$ . Then one of the following holds:*

- (i) For some  $\Delta \subset \Omega$  with  $|\Delta| < s$  we have  $G_{(\Delta)} \leq K \leq G_{\{\Delta\}}$ ;
- (ii)  $n = 2m$  is even,  $K$  is imprimitive with two blocks of size  $m$ , and  $|G : K| = \frac{1}{2} \binom{n}{m}$ ; or
- (iii) one of six exceptional cases hold where:
  - (a)  $K$  is imprimitive on  $\Omega$  and  $(n, s, |G : K|) = (6, 3, 15)$ ;
  - (b)  $K$  is primitive on  $\Omega$  and  $(n, s, |G : K|, K) = (5, 2, 6, 5 : 2), (6, 2, 6, PSL_2(5)), (7, 2, 15, PSL_3(2)), (8, 2, 15, AGL_3(2))$  or  $(9, 4, 120, P\Gamma L_2(8))$ .

**Remark 2.** (1) From part (i) of Lemma 2.5 we know that  $K$  contains the alternating group  $G_{(\Delta)} = \text{Alt}(\Omega \setminus \Delta)$  of degree  $n - s + 1$ .

(2) A result similar to Lemma 2.5 holds for the finite symmetric groups  $\text{Sym}(\Omega)$  which can be found in [5, Theorem 5.2B].

We will also need some elementary inequalities.

**Lemma 2.6.** *Let  $s$  and  $t$  be two positive integers.*

- (i) If  $t > s \geq 7$ , then  $\binom{s+t}{s} > 2t^4 > 2s^2t^2$ .
- (ii) If  $s \geq 6$  and  $t \geq 2$ , then  $2^{(s-1)(t-1)} > 2s^4 \binom{t}{2}^2$  implies  $2^{s(t-1)} > 2(s+1)^4 \binom{t}{2}^2$ .
- (iii) If  $t \geq 6$  and  $s \geq 2$ , then  $2^{(s-1)(t-1)} > 2s^4 \binom{t}{2}^2$  implies  $2^{(s-1)t} > 2s^4 \binom{t+1}{2}^2$ .
- (iv) If  $t \geq 4$ , and  $s \geq 3$ , then  $\binom{s+t}{s} > 2s^2t^2$  implies  $\binom{s+t+1}{s} > 2s^2(t+1)^2$ .

**Proof.** (i) It is necessary to prove that  $\binom{s+t}{s} > 2t^4$  holds. Since  $t > s \geq 7$  then  $\lfloor \frac{s+t}{2} \rfloor \geq s \geq 7$ , it follows that  $\binom{s+t}{s} \geq \binom{t+7}{7} > 2t^4$ .

(ii) Suppose that  $s \geq 6$ ,  $t \geq 2$  and  $2^{(s-1)(t-1)} > 2s^4 \binom{t}{2}^2$ . Then

$$2^{s(t-1)} = 2^{(s-1)(t-1)} 2^{t-1} > 2s^4 \binom{t}{2}^2 2^{t-1} = 2(s+1)^4 \binom{t}{2}^2 \left(1 - \frac{1}{s+1}\right)^4 2^{t-1}.$$

Combing this with the fact  $(1 - \frac{1}{s+1})^4 2^{t-1} \geq 2 \times (\frac{6}{7})^4 > 1$  gives (ii).

(iii) Suppose that  $t \geq 6$ ,  $s \geq 2$  and  $2^{(s-1)(t-1)} > 2s^4 \binom{t}{2}^2$ . Then

$$2^{(s-1)t} = 2^{(s-1)(t-1)} 2^{s-1} > 2s^4 \binom{t}{2}^2 2^{s-1} = 2s^4 \binom{t+1}{2}^2 \left(1 - \frac{2}{t+1}\right)^2 2^{s-1}.$$

Combing this with the fact  $(1 - \frac{2}{t+1})^2 2^{s-1} \geq 2 \times (\frac{5}{7})^2 > 1$  gives (iii).

(iv) Suppose that  $t \geq 4$ ,  $s \geq 3$  and  $\binom{s+t}{s} > 2s^2t^2$ . Then

$$\binom{s+t+1}{s} = \binom{s+t}{s} \frac{s+t+1}{t+1} > 2s^2t^2 \frac{s+t+1}{t+1} = 2s^2(t+1)^2 \frac{(s+t+1)t^2}{(t+1)^3}.$$

The fact that  $(s+t+1)t^2 > (t+1)^3$  gives (iv).

□

### 3 Proof of Theorem 1.1

In this section,  $\mathcal{D}$  denotes a non-trivial non-symmetric  $2$ -( $v, k, 2$ ) design if without special statement, and  $G \leq \text{Aut}(\mathcal{D})$  is flag-transitive point-primitive with  $\text{Soc}(G) = A_n$ . Let  $\alpha$  be a point of  $\mathcal{D}$  and  $H = G_\alpha$ . Since  $G$  is point-primitive,  $H$  is a maximal subgroup of  $G$  by [12, Theorem 8.2]. Furthermore, by the flag-transitivity of  $G$ , we have that  $v = |G : H|$ ,  $b \mid |G|$ ,  $r \mid |H|$  and  $r^2 > 2v$  by Lemma 2.2 (i).

If  $r$  is odd, then  $(r, 2) = 1$ . This case has been classified by Zhou and Wang in [11], and we get the following:

**Proposition 3.1.** *Let  $\mathcal{D}$  be a non-trivial non-symmetric  $2$ -( $v, k, 2$ ) design admitting a flag-transitive point-primitive automorphism group  $G$  with  $\text{Soc}(G) = A_n$ ,  $n \geq 5$ . If the replication number  $r$  is odd, then  $\mathcal{D}$  is the unique  $2$ -( $6, 3, 2$ ) design and  $G = A_5$ .*

Now we assume that  $r$  is even in the following.

Suppose first that  $n = 6$  and  $G \cong M_{10}$ ,  $PGL_2(9)$  or  $P\Gamma L_2(9)$ . Each of these groups has exactly three maximal subgroups with index greater than 2, and their indices are precisely 45, 36 and 10. By using the computer algebra system **GAP** [6], for  $v = 45, 36$  or 10, we will compute the parameters  $(v, b, r, k)$  that satisfy the following conditions:

$$r \mid (2(v-1), |H|); \tag{3.1}$$

$$r^2 > 2v; \tag{3.2}$$

$$2 \mid r; \tag{3.3}$$

$$r(k-1) = 2(v-1); \tag{3.4}$$

$$r > k > 2; \tag{3.5}$$

$$b = \frac{vr}{k}; \tag{3.6}$$

We obtain two possible parameters  $(v, b, r, k)$  as follows:

$$(10, 15, 6, 4) \text{ and } (36, 45, 10, 8).$$

Now we consider the existence of flag-transitive point-primitive non-symmetric designs with above possible parameters.

Suppose first that there exists a 2-(10, 4, 2) design  $\mathcal{D}$  with a flag-transitive point-primitive automorphism group  $G$ . Let the point set  $\mathcal{P} = \{1, 2, \dots, 10\}$ , and the group  $G = M_{10}$ ,  $PGL_2(9)$  or  $P\Gamma L_2(9)$  as the primitive permutation group of degree 10 acting on  $\mathcal{P}$ . Since  $G$  is flag-transitive,  $G$  acts block-transitively on  $\mathcal{B}$ , so  $|G|/b = |G_B|$ , where  $B$  is a block. For each case, using the command `Subgroups(G:OrderEqual:=n)` where  $n = |G|/b$  by **Magma** [1], it turns out that  $G$  has no subgroup of order  $n$ , which contradicts the fact that  $G_B$  is a subgroup of order  $|G|/b$ .

Assume next that there exists a 2-(36, 8, 2) design  $\mathcal{D}$  with the flag-transitive point-primitive automorphism group  $G = M_{10}$ ,  $PGL_2(9)$  or  $P\Gamma L_2(9)$ .

When  $(v, G) = (36, P\Gamma L_2(9))$ , by the **Magma**-command `Subgroups(G:OrderEqual:=n)` where  $n = |G|/b$ , we get the block stabilizer  $G_B$ . Since  $G$  is flag-transitive,  $G_B$  is transitive on  $B$ , and so  $B$  is an orbit of  $G_B$  acting on the point set  $\mathcal{P}$ . Using the **Magma**-command `Orbits(GB)` where  $GB = G_B$ , it turns out that  $G_B$  has no orbit of length  $k$ , a contradiction.

Now assume that  $(v, G) = (36, M_{10})$  or  $(36, PGL_2(9))$ . By the definition of 2-( $v, k, \lambda$ ) design, every pair of distinct points is contained in exactly  $\lambda$  blocks for some positive constant  $\lambda$ , i.e. *pairwise balanced*. However, for each case, the command `PairwiseBalancedLambda(D)` turns out false.

Take  $(v, G) = (36, M_{10})$  for example, the orbits of  $G_B$  are:

$$\Delta_0 = \{3, 17, 18, 21\},$$

$$\Delta_1 = \{1, 4, 12, 14, 16, 22, 26, 34\},$$

$$\Delta_2 = \{2, 6, 7, 9, 15, 23, 29, 36\},$$

$$\Delta_3 = \{5, 8, 10, 11, 13, 19, 20, 24, 25, 27, 28, 30, 31, 32, 33, 35\}.$$

As  $k = 8$ , we take the orbit of length 8 as the block  $B$ , i.e.  $B = \Delta_1$  or  $B = \Delta_2$ . Using the **GAP**-command `D1:=BlockDesign(36, [[1,4,12,14,16,22,26,34]], G)`, we obtain that  $|\Delta_1^G| = 45 = b$ . We take  $\mathcal{P} = \{1, 2, \dots, 36\}$ ,  $B = \Delta_1$  and  $\mathcal{B} = B^G$ . Now, we just need check that each pair of distinct points is contained in 2 blocks. However, `PairwiseBlancedLambda(D1)` shows that this is not true, and so  $B \neq \Delta_1$ . Similarly, we can get  $B \neq \Delta_2$ . So the case  $(v, G) = (36, M_{10})$  cannot occur.

Now we consider  $G = A_n$  or  $S_n$  with  $n \geq 5$ . The point stabilizer  $H = G_\alpha$  acts both on  $\mathcal{P}$  and the set  $\Omega_n = \{1, 2, \dots, n\}$ . Then by Lemma 2.3 one of the following holds:

- (i)  $H$  is primitive in its action on  $\Omega_n$ ;
- (ii)  $H$  is transitive and imprimitive in its action on  $\Omega_n$ ;
- (iii)  $H$  is intransitive in its action on  $\Omega_n$ .

We analyse each of these actions separately. First of all, we assume a hypothesis.

**Hypothesis 1.** *Let  $\mathcal{D}$  be a non-trivial non-symmetric  $2$ -( $v, k, 2$ ) design admitting a flag-transitive point-primitive automorphism group  $G$  with  $\text{Soc}(G) = A_n$  ( $n \geq 5$ ) and let the replication number  $r$  be even.*

### 3.1 $H$ acts primitively on $\Omega_n$

**Proposition 3.2.** *Assume that Hypothesis 1 holds and the point stabilizer  $H$  acts primitively on  $\Omega_n$ , then there exist 10 possible parameters  $(n, v, b, r, k)$  which are listed in Table 3.*

**Proof.** We claim that  $2 \parallel r$ . Otherwise  $4 \mid r$ , from the basic equation  $r(k-1) = 2(v-1)$ , we have that  $2 \mid (v-1)$ , and so  $v$  is odd. Thus by Lemma 2.4,  $v = 15$ ,  $G = A_7$  and  $|H| = |G|/v = 168$ . Since  $r \mid (2(v-1), |H|)$ ,  $r^2 > 2v$  and  $k \geq 3$ ,  $r = 7$  or  $14$ , which contradicts  $4 \mid r$ .

Thus  $2 \parallel r$ . Since  $r > 2$ , there exists an odd prime  $p$  that divides  $r$ , then  $p \mid (v-1)$ , and so  $(p, v) = 1$ . Thus  $H$  contains a Sylow  $p$ -subgroup  $P$  of  $G$ . Let  $g \in G$  be a  $p$ -cycle, then there is a conjugate of  $g$  belongs to  $H$ . This implies that  $H$  acting on  $\Omega_n$  contains an even permutation with exactly one cycle of length  $p$  and  $n-p$  fixed points. By a result of Jordan [12, Theorem 13.9], we have that  $n-p \leq 2$ . Therefore,  $n-2 \leq p \leq n$ ,  $p^2 \nmid |G|$ , and so  $p^2 \nmid r$ . It follows that  $r = 2(n-2)$ ,  $2(n-1)$ ,  $2n$  or  $2n(n-2)$ , where  $n$ ,  $n-1$  and  $n-2$  are odd primes. Moreover, the primitivity of  $H$  acting on  $\Omega_n$  and  $H \not\cong A_n$  imply that  $v \geq \frac{[n+1]}{2}!$  by [12, Theorem 14.2]. Combining with  $r^2 > 2v$ , we have that

$$r^2 > \left[\frac{n+1}{2}\right]!.$$

Therefore,  $(n, r) = (5, 6)$ ,  $(5, 10)$ ,  $(5, 30)$ ,  $(6, 10)$ ,  $(7, 10)$ ,  $(7, 14)$ ,  $(7, 70)$ ,  $(8, 14)$ ,  $(9, 14)$  or  $(13, 286)$ . By Lemmas 2.1 and 2.2, the facts  $v \geq \frac{[n+1]}{2}!$  and  $[b, v] \mid |G|$ , we obtain 10 possible parameters  $(n, v, b, r, k)$ :

$$\begin{aligned} &(5, 10, 15, 6, 4), (6, 16, 40, 10, 4), (6, 36, 45, 10, 8), (7, 15, 70, 14, 3), (7, 16, 40, 10, 4), \\ &(7, 21, 42, 10, 5), (7, 36, 45, 10, 8), (7, 36, 84, 14, 6), (8, 15, 70, 14, 3), (8, 36, 84, 14, 6). \end{aligned}$$

And we list them in Table 3.

□

### 3.2 $H$ acts transitively and imprimitively on $\Omega_n$

**Proposition 3.3.** *Assume that Hypothesis 1 holds and the point stabilizer  $H$  acts transitively but imprimitively on  $\Omega_n$ , then there exist 2 possible parameters  $(n, v, b, r, k) = (6, 10, 15, 6, 4)$  or  $(10, 126, 1050, 50, 6)$  which are listed in Table 3.*

**Proof.** Suppose on the contrary that  $\Sigma = \{\Delta_0, \Delta_1, \dots, \Delta_{t-1}\}$  is a non-trivial partition of  $\Omega_n$  preserved by  $H$ , where  $|\Delta_i| = s$ ,  $0 \leq i \leq t-1$ ,  $s, t \geq 2$  and  $st = n$ . Then

$$v = \binom{ts-1}{s-1} \binom{(t-1)s-1}{s-1} \cdots \binom{3s-1}{s-1} \binom{2s-1}{s-1}.$$

Moreover, the set  $O_j$  of  $j$ -cyclic partitions with respect to  $X$  (a partition of  $\Omega_n$  into  $t$  classes each of size  $s$ ) is an union of orbits of  $H$  on  $\mathcal{P}$  for  $j = 2, \dots, t$  (see [4, 13] for definitions and details).

**Case (1):** Suppose first that  $s = 2$ , then  $t \geq 3$ ,  $v = (2t-1)(2t-3) \cdots 5 \cdot 3$ , and

$$d_j = |O_j| = \frac{1}{2} \binom{t}{j} \binom{s}{1}^j = 2^{j-1} \binom{t}{j}.$$

We claim that  $t < 7$ . If  $t \geq 7$ , then  $v = (2t-1)(2t-3) \cdots 5 \cdot 3 > 5t^2(t-1)^2$ . On the other hand, since  $r$  divides  $2d_2 = 2t(t-1)$ ,  $2t(t-1) \geq r$ , and so  $v < 2t^2(t-1)^2$ , a contradiction. Thus  $t < 7$ . For  $t = 3, 4, 5$  or  $6$ , we calculate  $d = 2\gcd(d_2, d_3)$ , which are listed in Table 1 below.

Table 1: Possible  $d$  when  $s = 2$ .

$t$	$n$	$v$	$d_2$	$d_3$	$d$
3	6	15	6	4	4
4	8	105	12	16	8
5	10	945	20	40	40
6	12	10395	30	80	20

In each line  $r \leq d$  which contradicts the fact  $r^2 > 2v$ .

**Case (2):** Thus  $s \geq 3$ . So  $O_j$  is an orbit of  $H$  on  $\mathcal{P}$ , and  $d_j = |O_j| = \binom{t}{j} \binom{s}{1}^j = s^j \binom{t}{j}$ . In particular,  $d_2 = s^2 \binom{t}{2}$  and  $r \mid 2d_2$ . Moreover, from  $\binom{is-1}{s-1} = \frac{is-1}{s-1} \cdot \frac{is-2}{s-2} \cdots \frac{is-(s-1)}{1} > i^{s-1}$ , for  $i = 2, 3, \dots, t$ , we have that  $v > 2^{(s-1)(t-1)}$ . Then

$$2 \cdot 2^{(s-1)(t-1)} < 2v < r^2 \leq 4s^4 \binom{t}{2}^2,$$



and so

$$2^{(s-1)(t-1)} < 2s^4 \binom{t}{2}^2.$$

We will calculate all pairs  $(s, t)$  satisfying above inequality. Since  $2^{(6-1)(6-1)} = 2^{25} > 2 \cdot 6^4 \cdot \binom{6}{2}^2 = 2^5 \cdot 3^6 \cdot 5^2$ , the pair  $(s, t) = (6, 6)$  does not satisfy the inequality, but satisfies the conditions of Lemma 2.6 (ii) and (iii). Thus, we have either  $s < 6$  or  $t < 6$ . It is not hard to get 36 pairs  $(s, t)$  satisfying the inequality:

$$(3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (3, 7), (3, 8), (3, 9), (3, 10), (4, 2), (4, 3), (4, 4), (4, 5), \\ (4, 6), (5, 2), (5, 3), (5, 4), (5, 5), (6, 2), (6, 3), (6, 4), (7, 2), (7, 3), (8, 2), (8, 3), (9, 2), \\ (9, 3), (10, 2), (11, 2), (12, 2), (13, 2), (14, 2), (15, 2), (16, 2), (17, 2), (18, 2).$$

For each pair  $(s, t)$ , we calculate the parameters  $(v, b, r, k)$  satisfying Lemmas 2.1, 2.2,  $2 \mid r$  and  $r \mid 2d_2$ . There exist 2 possible parameters  $(n, v, b, r, k)$  corresponding to  $(s, t)$ :

$$(s, t) = (3, 2) \text{ with } (n, v, b, r, k) = (6, 10, 15, 6, 4), \\ (s, t) = (5, 2) \text{ with } (n, v, b, r, k) = (10, 126, 1050, 50, 6),$$

which are listed in Table 3.

□

### 3.3 $H$ acts intransitively on $\Omega_n$

**Proposition 3.4.** *Assume that Hypothesis 1 holds and the point stabilizer  $H$  acts intransitively on  $\Omega_n$ , then there exist 15 possible parameters  $(n, v, b, r, k)$  which are listed in Table 3.*

**Proof.** Since  $H$  acts intransitively on  $\Omega_n$ , we have  $H = (\text{Sym}(S) \times \text{Sym}(\Omega_n \setminus S)) \cap G$ , and without loss of generality, we may assume that  $|S| = s < \frac{n}{2}$  by Lemma 2.3 (i). By the flag-transitivity of  $G$ ,  $H$  is transitive on the blocks through  $\alpha$ , and so  $H$  fixes exactly one point in  $\mathcal{P}$ . Since  $H$  stabilizes only one  $s$ -subset of  $\Omega_n$ , we can identify the point  $\alpha$  with  $S$ . As the orbit of  $S$  under  $G$  consists of all the  $s$ -subsets of  $\Omega_n$ , we can identify  $\mathcal{P}$  with the set of  $s$ -subsets of  $\Omega_n$ . So  $v = \binom{n}{s}$ ,  $G$  has rank  $s + 1$  and the subdegrees are:

$$d_0 = 1, d_{i+1} = \binom{s}{i} \binom{n-s}{s-i}, i = 0, 1, 2, \dots, s-1.$$

We claim that  $s \leq 6$ . It follows from  $r \mid 2d_s$  and  $d_s = s(n-s)$  that  $r \mid 2s(n-s)$ . Combining this with  $r^2 > 2v$ , we have that  $2s^2(n-s)^2 > \binom{n}{s}$ . Since  $s < \frac{n}{2}$  equals to

$s < t = n - s$ , we have

$$2s^2t^2 > \binom{s+t}{s}.$$

Combining this with Lemma 2.6 (i), we have  $s \leq 6$ .

**Case (1):** If  $s = 1$ , then  $v = n \geq 5$  and the subdegrees are 1,  $n - 1$ . If  $k = v - 1$ , then  $r(v - 2) = 2(v - 1)$ , and so  $v - 2 \mid v - 1$  for  $(r, 2) = 2$ , a contradiction. Therefore,  $2 < k \leq v - 2$ . Since  $G$  is  $(v - 2)$ -transitive on  $\mathcal{P}$ ,  $G$  acts  $k$ -transitively on  $\mathcal{P}$ , and so  $b = |\mathcal{B}| = |B^G| = \binom{n}{k}$  for every block  $B \in \mathcal{B}$ . From the equality  $bk = vr$ , we obtain  $\binom{n}{k}k = nr$ . On one hand, by  $r(k - 1) = 2(n - 1)$  and  $k > 2$ , we have  $r \leq n - 1$ , and so  $\binom{n}{k}k \leq n(n - 1)$ ; on the other hand, by  $2 < k \leq n - 2$ , we have  $n - i \geq k - i + 2 > k - i + 1$  for  $i = 2, 3, \dots, k - 1$ . Thus,

$$\binom{n}{k}k = n(n - 1) \cdot \frac{n - 2}{k - 1} \cdot \frac{n - 3}{k - 2} \cdots \frac{n - k + 1}{2} > n(n - 1),$$

a contradiction.

**Case (2):** If  $s = 2$ , then  $v = \frac{n(n-1)}{2}$  and the subdegrees are 1,  $\binom{n-2}{2}$ ,  $2(n - 2)$ . By Lemma 2.2 (iii),  $r \mid 2(\binom{n-2}{2}, 2(n - 2))$ .

(a) If  $n \equiv 0 \pmod{4}$  or  $n \equiv 2 \pmod{4}$ , then  $r \mid 2(\binom{n-2}{2}, 2(n - 2)) = n - 2$ , it follows that  $n(n - 1) = 2v < r^2 \leq (n - 2)^2$ , which is impossible.

(b) If  $n \equiv 1 \pmod{4}$ , then  $r \mid 2(\binom{n-2}{2}, 2(n - 2)) = 2(n - 2)$ .

Let  $r = \frac{2(n-2)}{u}$  for some integer  $u$ . Since  $r^2 > 2v$ , we have  $4 > \frac{4(n-2)^2}{n(n-1)} > u^2$ , which forces  $u = 1$ . Therefore,  $r = 2(n - 2)$ . By Lemma 2.1,  $k = \frac{n+3}{2}$  and  $b = \frac{2n(n-1)(n-2)}{n+3}$ . Since  $k$  and  $b$  are integers,  $(n + 3) \mid 120$  with  $n$  odd. Combining this with  $n \equiv 1 \pmod{4}$ , we get that  $n = 5, 9, 17, 21, 37, 57$  or  $117$ . For each  $n$ , we calculate the parameters  $(v, b, r, k)$ . We find that if  $n \in \{17, 21, 37, 57, 117\}$ , then  $|G : G_B| = b < \binom{n}{3}$ . By Lemma 2.5 and [5, Theorem 5.2B], it is easy to know that  $G$  has no subgroup of index  $b$ , a contradiction. So we obtain 2 possible parameters  $(n, v, b, r, k)$ :

$$(5, 10, 15, 6, 4), (9, 36, 84, 14, 6).$$

(c) If  $n \equiv 3 \pmod{4}$ , then  $r \mid 2(\binom{n-2}{2}, 2(n - 2)) = 4(n - 2)$ .

Let  $r = \frac{4(n-2)}{u}$  for some integer  $u$ . Since  $r^2 > 2v$ , we have  $16 > \frac{16(n-2)^2}{n(n-1)} > u^2$ , and so  $u = 1, 2$  or  $3$ .

If  $u = 1$ , then  $r = 4(n - 2)$ ,  $k = \frac{n+5}{4}$  and  $b = \frac{8n(n-1)(n-2)}{n+5}$ . As  $b$  must be an integer,  $(n + 5) \mid 1680$ . Combining with  $n \equiv 3 \pmod{4}$ , we have that  $n = 7, 11, 15, 19, 23, 35, 43, 51, 55, 75, 79, 107, 115, 135, 163, 235, 275, 331, 415, 555, 835$  or  $1675$ . Similarly, by Lemma 2.5 and [5, Theorem 5.2B],  $n \in \{7, 11, 15, 19, 23, 35, 43\}$  and we obtain 7 possible parameters

$(n, v, b, r, k)$ :

$$(7, 21, 140, 20, 3), (11, 55, 495, 36, 4), (15, 105, 1092, 52, 5), (19, 171, 1938, 68, 6), \\ (23, 253, 3036, 84, 7), (35, 595, 7854, 132, 10), (43, 903, 12341, 164, 12).$$

If  $u = 2$ , then  $r = 2(n-2)$ ,  $k = \frac{n+3}{2}$  and  $b = \frac{2n(n-1)(n-2)}{n+3}$ , and so  $(n+3) \mid 120$ . Combining with  $n \equiv 3 \pmod{4}$ , we have that  $n = 7$  or  $27$ . By Lemma 2.5 and [5, Theorem 5.2B],  $n \neq 27$ , so we obtain a possible parameter  $(n, v, b, r, k) = (7, 21, 42, 10, 5)$ .

If  $u = 3$ , then  $r = \frac{4(n-2)}{3}$ ,  $k = \frac{3n+7}{4}$  and  $b = \frac{8n(n-1)(n-2)}{3(3n+7)}$ , and so  $(3n+7) \mid 7280$ . The facts that  $r$  is an integer and  $n \equiv 3 \pmod{4}$  imply  $n = 11, 35, 119$  or  $1211$ . For each  $n$ , we find  $|G : G_B| = b < \binom{n}{3}$ . By Lemma 2.5 and [5, Theorem 5.2B], it is easy to know that  $G$  has no subgroup of index  $b$ , a contradiction.

**Case (3):** Suppose that  $3 \leq s \leq 6$ . Now for each value of  $s$ , by the inequality  $2s^2t^2 > \binom{s+t}{s}$  and Lemma 2.6 (iv), we know that  $t$  (and hence  $n$ ) is bounded. For example, let  $s = 3$ , since  $\binom{3+102}{3} > 2 \cdot 3^2 \cdot 102^2$ , we must have  $4 \leq t \leq 101$ , and so  $7 \leq n \leq 104$ . The bounds of  $n$  are listed in Table 2 below.

Table 2: Bounds of  $n$  when  $3 \leq s \leq 6$

$s$	$t$	$n$
3	$4 \leq t \leq 101$	$7 \leq n \leq 104$
4	$5 \leq t \leq 22$	$9 \leq n \leq 26$
5	$6 \leq t \leq 12$	$11 \leq n \leq 17$
6	7, 8, 9	13, 14, 15

Note that  $v = \binom{n}{s}$ , and  $d_1 = \binom{n-s}{s}$ ,  $d_2 = s \binom{n-s}{s-1}$ ,  $d_3 = \binom{s}{2} \binom{n-s}{s-2}$  are three non-trivial subdegrees of  $G$  acting on  $\mathcal{P}$ . Therefore, the 5-tuple  $(n, v, b, r, k)$  satisfies the arithmetical conditions: (3.1)-(3.6) and  $r \mid 2d_i$ ,  $i \in \{1, 2, 3\}$ .

If  $s = 3$ , using **GAP**, it outputs five 5-tuples:

$$(13, 286, 429, 30, 20), (14, 364, 2002, 66, 12), (22, 1540, 6270, 114, 28), \\ (32, 4960, 14880, 174, 58), (50, 19600, 39480, 282, 140).$$

If  $s = 4, 5$  or  $6$ , using **GAP**, there is no parameter  $(n, v, b, r, k)$  satisfying these conditions. Thus, we totally obtain 15 possible parameters  $(n, v, b, r, k)$  and list them in Table 3.

□

Table 3: Potential parameters

CASE	$(v, b, r, k)$	$\text{Soc}(G)$ or $G$	Proposition	Step/Reference
1	(10, 15, 6, 4)	$A_5$	3.2	(ii)
2		$A_6$	3.3	$\mathcal{D}$
3		$G = A_5$	3.4	(v)
4		$G = S_5$	3.4	$\mathcal{D}$
5	(15, 70, 14, 3)	$G = A_7$ or $A_8$	3.2	(v)
6		$G = S_7$ or $S_8$	3.2	(i)
7	(16, 40, 10, 4)	$A_6, A_7$	3.2	(i)
8	(21, 42, 10, 5)	$A_7$	3.2	(ii)
9		$A_7$	3.4	(vi)
10	(21, 140, 20, 3)	$A_7$	3.4	(vi)
11	(36, 45, 10, 8)	$A_6, A_7$	3.2	(i)
12	(36, 84, 14, 6)	$A_7, A_8$	3.2	(i)
13		$A_9$	3.4	(iv)
14	(55, 495, 36, 4)	$A_{11}$	3.4	(iv)
15	(105, 1092, 52, 5)	$A_{15}$	3.4	(iii)
16	(126, 1050, 50, 6)	$A_{10}$	3.3	(iii)
17	(171, 1938, 68, 6)	$A_{19}$	3.4	(iv)
18	(253, 3036, 84, 7)	$A_{23}$	3.4	(iii)
19	(286, 429, 30, 20)	$A_{13}$	3.4	(iii)
20	(364, 2002, 66, 12)	$A_{14}$	3.4	(iv)
21	(595, 7854, 132, 10)	$A_{35}$	3.4	(iii)
22	(903, 12341, 164, 12)	$A_{43}$	3.4	(iv)
23	(1540, 6270, 114, 28)	$A_{22}$	3.4	(iii)
24	(4960, 14880, 174, 58)	$A_{32}$	3.4	(vii)
25	(19600, 39480, 282, 140)	$A_{50}$	3.4	(iii)

### 3.4 Rules out potential parameters

Now, we will rule out 23 potential cases listed in Table 3 in several steps.

(i) Rules out CASES 6, 7, 11 and 12.

The GAP-command `PrimitiveGroup(v,nr)` returns the primitive group with degree  $v$  in the position  $nr$  in the list of the library of the primitive permutation groups. For each CASE, the command shows that there is no primitive group corresponding to  $v$ .

(ii) Rules out CASES 1 and 8.

Since  $G$  is flag-transitive,  $|H| = |G|/v$ . For each case,  $H$  is primitive on  $\Omega_n$ . However, the command `PrimitiveGroup(v,nr)`, where  $v = n$ , turns out that there is no group of order  $|G|/v$ .

(iii) Rules out CASES 15, 16, 18, 19, 21, 23 and 25.

Since  $G$  is flag-transitive,  $G$  acts transitively on  $\mathcal{B}$ , so  $|G|/b = |G_B|$ , where  $B$  is a block. For each case, using the Magma-command `Subgroups(G:OrderEqual:=n)` where  $n = |G|/b$ , it turns out that  $G$  has no subgroup of order  $n$ , which contradicts the fact that  $G_B$  is a subgroup of order  $|G|/b$ . When  $v \geq 2500$ , the GAP-command `PrimitiveGroup(v,nr)` does not know the group of degree  $v$ . For CASE 25,  $G = A_{50}$  or  $S_{50}$ , we use the Magma-command  $G := \text{Alt}(50)$  or  $G := \text{Sym}(50)$  to get the group  $G$ , and `Subgroups(G:OrderEqual:=n)` where  $n = |G|/b$  to get that  $G$  does not have such a subgroup of order  $|G|/b$ , a contradiction.

(iv) Rules out CASES 13, 14, 17, 20 and 22.

Since  $G_B$  is transitive on  $B$ ,  $B$  is an orbit of  $G_B$  acting on the point set  $\mathcal{P}$ . Using the Magma-command `Orbits(GB)`, where  $GB = G_B$ , it turns out that  $G_B$  has no orbit of length  $b$ , a contradiction.

(v) Rules out CASES 3 and 5.

Using the command `Orbits(GB)`, we get the orbits of  $G_B$ . As  $|B| = k$ , we take the orbit of length  $k$  as the block  $B$ . Since  $G$  acts transitively on  $\mathcal{B}$ ,  $|B^G| = b$ . However, for each case, using the GAP-command `OrbitLength(G,B,OnSets)`, we get that  $|B^G| < b$ , a contradiction.

Take CASE 3,  $(v, G) = (10, A_5)$  for example, the orbits of  $G_B$  are:

$$\Delta_0 = \{1, 8\}, \quad \Delta_1 = \{2, 6\}, \quad \Delta_2 = \{3, 5\}, \quad \Delta_3 = \{4, 7, 9, 10\}.$$

So  $B = \Delta_3 = \{4, 7, 9, 10\}$ . However, `OrbitLength(G, [4, 7, 9, 10], OnSets)` turns out  $|B^G| = 5 < 15$ , a contradiction.

(vi) Rules out CASES 9 and 10.

For each case, the GAP-command `PairwiseBalancedLambda(D)` turns out false, which means that  $D$  is not pairwise balanced, contradicting the definition of design.

For CASE 9, take  $(v, G) = (21, A_7)$  for example, the orbits of  $G_B$  are:

$$\begin{aligned} \Delta_0 &= \{13\}, & \Delta_1 &= \{2, 7, 12, 14, 15\}, \\ \Delta_2 &= \{4, 9, 16, 19, 20\}, & \Delta_3 &= \{1, 3, 5, 6, 8, 10, 11, 17, 18, 21\}. \end{aligned}$$

As  $k = 5$ , we take the orbit of length 5 as the block  $B$ , i.e.  $B = \Delta_1$  or  $B = \Delta_2$ . Using the GAP-command  $D := \text{BlockDesign}(21, [[2, 7, 12, 14, 15]], G)$ , we obtain that  $|\Delta_1^G| = 42$  and  $\Delta_2 \in \Delta_1^G$ . Without loss of generality, we take  $\mathcal{P} = \{1, 2, \dots, 21\}$ ,  $B = \Delta_1$  and  $\mathcal{B} = \Delta_1^G$ . Now, we just need check that  $D$  is pairwise balanced. However, `PairwiseBalancedLambda(D)` shows that this is not true. So the case  $(v, G) = (21, A_7)$  cannot occur.

(vii) Rules out CASE 24.

Consider first that  $(v, G) = (4960, A_{32})$ . Let  $\Omega_n = \{1, 2, \dots, 32\}$ , then  $G$  acts primitively on  $\Omega_n$ . Let  $\mathcal{P} = \Omega_n^{\{3\}}$  denote the set of all 3-subsets (that is, subsets of size 3) of  $\Omega_n$ .

Then  $G$  acts on  $\mathcal{P}$  in a natural way, namely:  $(\alpha_1, \alpha_2, \alpha_3)^g = (\alpha_1^g, \alpha_2^g, \alpha_3^g)$  for all  $g \in G$  and  $|\mathcal{P}| = \binom{32}{3} = 4960$ . Using the **Magma**-command  $\mathbf{G} := \mathbf{Alt}(32)$  to get the group  $G$ , and  $\mathbf{Subgroups}(\mathbf{G}:\mathbf{OrderEqual}:=\mathbf{n})$  where  $\mathbf{n} = |G|/b$  to know that  $G$  contains only one conjugacy class of subgroups of order  $|G|/b$ , denoted by  $K$ , so the block stabilizer  $G_B$  is conjugate to  $K$ , and then there is a block  $B_0$  such that  $K = G_{B_0}$ . Since  $G$  is flag-transitive,  $K$  is transitive on  $B_0$ , that is,  $B_0$  is an orbit of  $K$  acting on  $\mathcal{P}$ . Take  $S = \{1, 2, 3\} \in \mathcal{P}$ . Using the command  $\mathbf{OrbitLength}(\mathbf{G}, \mathbf{S}, \mathbf{OnSets})$ , we have that  $G$  acts transitively on  $\mathcal{P}$ , and using the command  $\mathbf{OrbitLength}(\mathbf{K}, \mathbf{S}', \mathbf{OnSets})$  for all  $\mathbf{S}' \in \mathcal{P}$ , we get that  $K$  acting on  $\mathcal{P}$  has exactly only one orbit  $\Gamma$  of length 58. As  $k = 58$ , we take  $B_0 = \Gamma$ . Furthermore, the **Magma**-command  $\mathbf{O} := \Gamma^\mathbf{G}$  turns out that  $|\mathbf{O}| = 14880 = b$ , and then we take  $\mathcal{B} = \mathbf{O}$ . Now, we just need check that each pair of distinct points is contained in 2 blocks. Let  $S_1 = \{1, 2, 3\}$ ,  $S_2 = \{5, 6, 9\} \in \mathcal{P}$ . Using **Magma**, it is easy to know that there is no block in  $\mathcal{B}$  containing both  $S_1$  and  $S_2$ , a contradiction. So the case  $(v, G) = (4960, A_{32})$  cannot occur.

The analysis of  $(v, G) = (4960, S_{32})$  is the same as above. Let  $\mathcal{P} = \Omega_n^{\{3\}}$  and  $G$  acts on  $\mathcal{P}$  in the natural way. Using **GAP** and **Magma**, we get the group  $G$ , the subgroup  $K = G_{B_0}$  of order  $|G|/b$ , and the orbit  $\Gamma$  ( $|\Gamma| = 58$ ) of  $K$  acting on  $\mathcal{P}$ . Since  $G$  is flag-transitive,  $G$  acts transitively on  $\mathcal{B}$ , and so  $|\Gamma^G| = 14880$ . Using the **Magma**-command

$$\mathbf{M} := \mathbf{PermutationGroup} < 32|(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11)(12, 13, 14, 15, 16, 17, 18, 19, \\ 20, 21, 22, 23, 24)(25, 26, 27), (9, 11, 15), (4, 13, 21) >,$$

we get a subgroup  $M < G$ , and so  $|\Gamma^M| \leq 14880$ . However,  $\mathbf{O} := \Gamma^\mathbf{M}$  turns out that  $|\mathbf{O}| = 36432 > 14880$ , a contradiction. So the case  $(v, G) = (4960, S_{32})$  is ruled out. □

### 3.5 The unique non-symmetric 2-(10, 4, 2) design

For CASE 2,  $(v, G) = (10, A_6)$ , using the **GAP**-command  $\mathbf{PrimitiveGroup}(10, 3)$ , we get the primitive permutation representations of  $G = A_6 \cong PSL(2, 9)$  acting on the set  $\mathcal{P} = \{1, 2, \dots, 10\}$ . By **Magma**, it is easy to know that  $G$  contains two conjugacy classes of subgroups of order 24, denoted by  $K_1$  and  $K_2$  as representatives. As  $G$  acts flag-transitively, for any  $B \in \mathcal{B}$ ,  $|G_B| = |G|/b = 24$ , so  $G_B$  is conjugate to either  $K_1$  or  $K_2$ .

Assume first that  $G_B$  is conjugate to  $K_1$ , then there exists a block  $B_0$  such that  $K_1 = G_{B_0}$ . By the flag-transitivity of  $G$ ,  $B_0$  is an orbit of  $K_1$  acting on  $\mathcal{P}$ . The **Magma**-command  $\mathbf{Orbits}(\mathbf{K1})$ , where  $\mathbf{K1} = K_1$ , turns out that the lengths of the orbits of  $K_1$  are 4 and 6. As  $|B_0| = k = 4$ , we take the orbit of length 4 as the block  $B_0$ . Using **GAP**, we obtain that

$|B_0^G| = 15$  and every pair of distinct points is contained in exactly 2 blocks. Therefore, we get a point-primitive non-symmetric  $2$ -( $10, 4, 2$ ) design, denoted by  $\mathcal{D}_1$ . Since  $G$  acts transitively on  $\mathcal{B}$  and  $G_{B_0}$  acts transitively on  $B_0$ ,  $G$  acts flag-transitively on  $\mathcal{D}_1$ . Thus, we get a desired flag-transitive point-primitive  $2$ -( $10, 4, 2$ ) non-symmetric design  $\mathcal{D}_1$ .

Consider next that  $G_B$  is conjugate to  $K_2$ . By **Magma**, we obtain that the orbits of  $K_2$  are  $\{2, 6, 7, 10\}$  and  $\{1, 3, 4, 5, 8, 9\}$ , and so we take  $B_0 = \{2, 6, 7, 10\}$ . Similarly, by the facts that  $|B_0^G| = 15$  and every pair of distinct points is contained in exactly 2 blocks, we get a flag-transitive point-primitive  $2$ -( $10, 4, 2$ ) non-symmetric design  $\mathcal{D}_2$ .

The analysis of CASE 2,  $(v, G) = (10, S_6)$  is the same as above, here the group  $G$  is **PrimitiveGroup**( $10, 5$ ). There are two conjugacy classes of subgroups of order 48, denoted by  $M_1$  and  $M_2$  as representatives. The orbits of  $M_1$  are  $\{2, 4, 6, 8\}$  and  $\{1, 3, 5, 7, 9, 10\}$ , and so we take  $B_0 = \{2, 4, 6, 8\}$ . Using **GAP**, we get that  $|B_0^G| = 15$  and every pair of distinct points is contained in exactly 2 blocks, and so we obtain a  $2$ -( $10, 4, 2$ ) design  $\mathcal{D}_3$  as desired. The lengths of the orbits of  $M_2$  are 4 and 6, and so we take the orbit of size 4 as the block  $B_0$ , which satisfies  $|B_0^G| = 15$  and  $\lambda = 2$ , hence, we get a desired design  $\mathcal{D}_4$ .

For CASE 4, we get the primitive group  $G = S_5$  by **PrimitiveGroup**( $10, 2$ ). Using the command **Subgroups(G:OrderEqual:=n)** where  $n = |G|/b$ , we know that there is exactly only one conjugacy class of subgroups of order 8, denoted by  $L$ , so the block stabilizer  $G_B$  is conjugate to  $L$ , and then there is a block  $B_0$  such that  $L = G_{B_0}$ . **Orbits(L)** gives the orbits of  $L$ :

$$\Delta_0 = \{3, 5\}, \quad \Delta_1 = \{1, 2, 6, 8\}, \quad \Delta_2 = \{4, 7, 9, 10\}.$$

So either  $B_0 = \Delta_1$  or  $B_0 = \Delta_2$ . If  $B_0 = \Delta_2$ , **OrbitLength(G, [4, 7, 9, 10], OnSets)** shows that  $|B_0^G| = 5 < 15$ , a contradiction. If  $B_0 = \Delta_1$ , **D := BlockDesign(10, [[1, 2, 6, 8]], G)** shows that  $|B_0^G| = 15$ , and **PairwiseBalancedLambda(D)** gives  $\lambda = 2$ . We take  $\mathcal{P} = \{1, 2, \dots, 10\}$ ,  $B_0 = \Delta_1$  and  $\mathcal{B} = B_0^G$ . Therefore, we construct a non-symmetric  $2$ -( $10, 4, 2$ ) design  $\mathcal{D}_5$ , which is flag-transitive and point-primitive.

The **GAP**-command **IsIsomorphicBlockDesign(D1, D2)** turns out true when  $D1 = \mathcal{D}_1$  and  $D2 = \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$  or  $\mathcal{D}_5$ . So up to isomorphism we take these five designs as the same design, denoted by  $\mathcal{D}$ .

This completes the proof of Theorem 1.1.

□

## References

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